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Neutral Functional Differential Equations with Infinite Delay

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This paper discusses the fundamental theory of neutral functional differential equations with infinite delay in the abstract phase spaces given in [1], [6]. Retarded functional differential equations with infinite delay and neutral equations with finite delay discussed by [1]–[3] are included in this class of equations.

In Section 1 we state the axioms for the phase space. Section 2 of this paper gives the definition of neutral functional differential equations with infinite delay and some examples. The main results of Section 3 concern the fundamental theory (existence, uniqueness, continuation of solutions) for this class of equations. In Section 4, we discuss the continuation of solution for a special kind of equations.

§ 1. Axioms for the phase space

Let $\hat{B}$ be a linear real vector sapce of functions mapping $(-\infty, 0]$ into $\boldsymbol{R}^n$ with elements denoted by $\hat{\varphi}, \hat{\psi} \cdots$, where $\hat{\varphi}=\hat{\psi}$ means $\hat{\varphi}(t)=\hat{\psi}(t)$ for $t\leq 0$. Assume that a seminorm $|\cdot|_{\hat{B}}$ is given in $\hat{B}$ so that $B=\hat{B}/_{|\cdot|}$ is a Banach space with the induced norm $|\varphi|_B=|\hat{\varphi}|_{\hat{B}}$ for $\varphi \in B$, $\hat{\varphi} \in \hat{B}$ if $\hat{\varphi} \in \varphi$.

For $\hat{x}$: $(-\infty, \sigma)\rightarrow \boldsymbol{R}^n$, $t \in (-\infty, \sigma)$, we define $\hat{x}_t$: $(-\infty, 0]\rightarrow \boldsymbol{R}^n$ by $\hat{x}_t(s)=\hat{x}(t+s)$ for $s\leq 0$. For $\alpha\geq 0$, $t_0 \in \boldsymbol{R}$ and $\hat{\varphi} \in \hat{B}$, let $\mathscr{F}_{\alpha,t_0}(\hat{\varphi})$ be the set of all functions $\hat{x}$: $(-\infty, t_0+\alpha]\rightarrow \boldsymbol{R}^n$ with $\hat{x}_{t_0}=\hat{\varphi}$ and $\hat{x}$ being continuous on $[t_0, t_0+\alpha]$ (on $[t_0, \infty)$ in case $\alpha=\infty$). Furthermore we put $\mathscr{F}_{\alpha,t_0}=\bigcup_{\hat{\varphi}\in\hat{B}} \mathscr{F}_{\alpha,t_0}(\hat{\varphi})$.

In case $t_0=0$ we simply write $\mathscr{F}_{\alpha}(\hat{\varphi})$ and $\mathscr{F}_{\alpha}$.

The first axiom for the phase space is

(α_1) $\hat{x}_t \in \hat{B}$ for $\hat{x} \in \mathscr{F}_{\alpha}$ and $t \in [0, \alpha]$.

For $\beta\geq 0$ and $\hat{\varphi} \in \hat{B}$, let $\hat{\varphi}^{\beta}$ denote the restriction of $\hat{\varphi}$ to $(-\infty, -\beta)$. For $\varphi \in B$, define

$$|\varphi|_{\beta}=\inf\{|\hat{\psi}|_{\hat{B}};\ \hat{\psi} \in \hat{B} \text{ and } \hat{\psi}^{\beta}=\hat{\varphi}^{\beta} \text{ for some } \hat{\varphi} \in \varphi\}.$$

$B^{\beta}=B_{|\cdot|_{\beta}}$ is the space of all equivalent classes $\{\varphi\}_{\beta}=\{\psi \in B;\ |\varphi-\psi|_{\beta}=0\}$ for $\varphi \in B$ with respect to the seminorm $|\cdot|_{\beta}$. In B^{β}, we define the norm $|\cdot|_{\beta}$ naturally induced by the seminorm $|\cdot|_{\beta}$.

For $\beta\geq 0$ and $\hat{\varphi} \in \hat{B}$, define

$$(\hat{S}_\beta\hat{\varphi})(\theta)=\begin{cases}\hat{\varphi}(\beta+\theta), & \theta \in (-\infty, -\beta),\\ \hat{\varphi}(0), & \theta \in [-\beta, 0].\end{cases}$$

The next axiom for the phase space is

(α_{2a}) if $|\hat{\varphi}-\hat{\psi}|_{\hat{B}}=0$, then $|\hat{S}_\beta\hat{\varphi}-\hat{S}_\beta\hat{\psi}|_{\hat{B}}=0$.

This axiom justifies the definition of S_β given by $S_\beta\varphi=\psi$ if $\hat{S}_\beta\hat{\varphi} \in \psi$ for $\hat{\varphi} \in \varphi$.

The other axioms are

(α_{2b}) if $\varphi=\psi$ in B, then $|S_\beta\varphi-S_\beta\psi|_\beta=0$ for $\beta\geq 0$,

(α_3') there exists a positive constant K such that for any $\hat{\varphi} \in \hat{B}$, $|\hat{\varphi}(0)|\leq K|\hat{\varphi}|_{\hat{B}}$.

The axiom (α_{2b}) justifies the definition of τ_β given by $\tau^\beta\varphi=\{S_\beta\varphi\}_\beta$ for $\varphi \in B$, while by the axiom (α_3') we can put $\varphi(0)=\hat{\varphi}(0)$ for $\hat{\varphi} \in \varphi$, and ($\alpha_3'$) is equivalent to

(α_3) there exists a positive constant K such that for any $\varphi \in B$, $|\varphi(0)|\leq K|\varphi|_B$.

The following is a key axiom for us to obtain the fundamental theory of functional differential equations defined in $\boldsymbol{R}\times B$.

(α_4') there exist a continuous function $K_1(s)$ and a locally bounded function $M_1(s)$ such that

(i) $|\tau^\beta\varphi|_\beta\leq M_1(\beta)|\varphi|_B$ for $\beta\geq 0$, $\varphi \in B$,

(ii) if $\hat{x} \in \mathscr{F}_{\alpha,t_0}$ then for $t \in [t_0, t_0+\alpha]$, we have

$$|\hat{x}_t|_{\hat{B}}\leq K_1(t-t_0)\sup_{t_0\leq s\leq t}|\hat{x}(s)|+M_1(t-t_0)|\hat{x}_{t_0}|_{\hat{B}}.$$

We define $\hat{z}$, $\hat{x} \in \mathscr{F}_{\alpha,t_0}$ to be equivalent, $\hat{z}\sim\hat{x}$, if and only if $|\hat{z}_{t_0}-\hat{x}_{t_0}|_{\hat{B}}=0$ and $\hat{z}(s)=\hat{x}(s)$ for $s \in [t_0, t_0+\alpha]$. The equivalent class of $\hat{z} \in \mathscr{F}_{\alpha,t_0}$ under "$\sim$" is denoted by z. Therefore we define for $\varphi \in B$, $t_0 \in \boldsymbol{R}$ and $\alpha\geq 0$

$$F_{\alpha,t_0}(\varphi)=\{z;\, \hat{z} \in \mathscr{F}_{\alpha,t_0}(\hat{\varphi}) \text{ for } \hat{z} \in z,\ \hat{\varphi} \in \varphi\}$$

and $F_{\alpha,t_0}=\bigcup_{\varphi \in B}F_{\alpha,t_0}(\varphi)$. Again we write $F_\alpha(\varphi)$ and F_α in case $t_0=0$. For $x \in F_{\alpha,t_0}$, $\hat{x} \in x$ and $t \in [t_0, t_0+\alpha]$ we can define $x_t=\{\hat{x}_t\}$ by (ii) of (α_4'), and define $x(t)=\hat{x}(t)$ by the definition of "$\sim$".

Then, (α_4') can be written as

(α_4) there exist a continuous function $K_1(s)$ and a locally bounded function $M_1(s)$ such that

(i) $|\tau^\beta\varphi|_\beta\leq M_1(\beta)|\varphi|_B$ for $\beta\geq 0$, $\varphi \in B$,

(ii) if $x \in F_{\alpha,t_0}$, then for $t \in [t_0, t_0+\alpha]$, we have

$$|x_t|_B\leq K_1(t-t_0)\sup_{t_0\leq s\leq t}|x(s)|+M_1(t-t_0)|x_{t_0}|_B.$$

The final axiom is

(α_5) if $x \in F_\alpha$, $\alpha>0$, then x_t is continuous in $t \in [0, \alpha]$.

§ 2. The definitions of NFDE and some examples

Definition 1. Suppose that Ω is an open set in $\boldsymbol{R}\times B$, D: $\Omega\to\boldsymbol{R}^n$ is continuous, $D(t, \varphi)$ has a continuous Fréchet derivative $D_\varphi(t, \varphi)$ with respect to φ on Ω and

$$(1) \qquad D_\varphi(t, \varphi)\psi = A(t, \varphi)\psi(0) + L(t, \varphi, \psi)$$

for $(t, \varphi) \in \Omega$, $\psi \in B$. If $A(t, \varphi)$ is an $n\times n$ matrix such that $\det A(t, \varphi)\neq 0$ and $A(t, \varphi)$, $A^{-1}(t, \varphi)$ are continuous, and if $L(t, \varphi, \psi)$ is linear with respect to ψ and satisfies:

(H_1) there are an $\alpha_0>0$ and a continuous map $r(t, \varphi, \alpha)$: $\Omega\times[0, \alpha_0]\to\boldsymbol{R}^+$, $r(t, \varphi, 0)=0$, such that for $\psi \in B$ satisfying $|\psi|_\alpha=0$,

$$(2) \qquad |L(t, \varphi, \psi)|\leq r(t, \varphi, \alpha)|\psi|_B,$$

then we say D is generalized atomic at zero on Ω.

Definition 2. Suppose $D, f \in C(\Omega, \boldsymbol{R}^n)$ and that D is generalized atomic at zero on Ω. Then we say

$$(3) \qquad \frac{d}{dt}D(t, x_t) = f(t, x_t)$$

is a neutral functional differential equation with infinite delay (hereafter called NFDE (D, f, Ω)).

By a solution of (3) we mean an $x \in F_{A,\sigma}$ for some $A>0$ and $-\infty<\sigma<+\infty$ such that

(i) $(t, x_t) \in \Omega$ for $t \in [\sigma, \sigma+A]$,

(ii) $D(t, x_t)$ is continuously differentiable and satisfies (3) on $[\sigma, \sigma+A]$.

If, in addition, $x_\sigma=\varphi$, then we say x is a solution of (3) through (σ, φ) and we denote it by $x(t; \sigma, \varphi)$.

Example 1. Suppose k: $(-\infty, 0]\to(0, \infty)$ is continuous, nondecreasing, integrable on $(-\infty, 0)$ and such that $k(u+v)\leq k(u)k(v)$. B_k^1 represents the set of classes of equivalent maps from $(-\infty, 0)$ into $\boldsymbol{R}^n$ such that they are strongly measurable on $(-\infty, 0]$ and continuous on $[-r, 0]$, $r>0$, and

$$|\varphi| = \sup_{u\in[-r,0]}|\varphi(u)| + \int_{-\infty}^{-r} k(u)|\varphi(u)|du < +\infty.$$

By [4], the dual space $(B_k^1)^*$ consists of all ψ; $(-\infty, 0]\to\boldsymbol{R}^n$ such that the restriction of ψ to $(-\infty, -r)$ belongs to $L^\infty((-\infty, -r), \boldsymbol{R}^n)$, while the restriction to $[-r, 0]$ is of bounded variation, left continuous on $(-r, 0)$ and $\psi(0)=0$. The duality paring between $\varphi \in B_k^1$ and $\psi \in (B_k^1)^*$ is given by

$$\langle\psi,\varphi\rangle=\int_{-\infty}^{-r}\psi(u)\varphi(u)k(u)du+\int_{-r}^{0}[d\psi(u)]\varphi(u)$$

with $\psi\varphi$ and $[d\psi]\varphi$ standing for scalar products in the Euclidean space.

So, if D: $B_k^1\to\boldsymbol{R}^n$ is a bounded linear functional, then there exists η_D: $(-\infty, 0]\to\mathscr{L}(\boldsymbol{R}^n, \boldsymbol{R}^n)$, a matrix whose columns are in $(B_k^1)^*$ such that

$$D\varphi=\int_{-\infty}^{-r}k(s)\eta_D(s)\varphi(s)ds+\int_{-r}^{0}[d\eta_D(s)]\varphi(s).$$

Therefore, D induces a bounded linear map D^r: $C([-r, 0], \boldsymbol{R}^n)\to\boldsymbol{R}^n$:

$$D^r\varphi=\int_{-r}^{0}[d\eta_D(s)]\varphi(s) \qquad \text{for } \varphi\in C([-r,\ 0], \boldsymbol{R}^n).$$

If D^r: $C([-r, 0], \boldsymbol{R}^n)\to\boldsymbol{R}^n$ is atomic at zero (see [2], [3]) (i.e., $A=\eta_D(0)-\eta_D(0-)$ is nonsingular and $\mathrm{Var}_{[-s,0]}\bar{\eta}_D\to 0$ as $s\to 0$, where

$$\bar{\eta}_D(s)=\begin{cases}\eta_D(s), & -r\le s<0\\ \eta_D(0-) & s=0),\end{cases}$$

then $(d/dt)D(x_t)=f(t, x_t)$ is a NFDE.

Example 2. $C_\infty^\gamma=\{\varphi\in C((-\infty, 0], \boldsymbol{R}^n);\ e^{\gamma\theta}\varphi(\theta)\to\text{const. as } \theta\to-\infty\}$,

$$|\varphi|_{C_\infty^\gamma}=\sup_{\theta\le 0}e^{\gamma\theta}|\varphi(\theta)| \qquad \text{for } \varphi\in C_\infty^\gamma.$$

Suppose $D_\varphi(t, \varphi)\psi=L_1(t, \varphi, {}_r\psi)+L_2(t, \varphi, {}^r\psi)$ for some constant $r>0$, where $L_1(t, \varphi, \psi)$, $L_2(t, \varphi, \psi)$: $\Omega\times C_\infty^\gamma\to\boldsymbol{R}^n$ are bounded linear functionals with respect to ψ and

$${}_r\psi(s)=\begin{cases}\psi(s), & s<-r\\ \psi(-r), & -r\le s\le 0,\end{cases}$$

$${}^r\psi(s)=\begin{cases}\psi(-r)e^{-\gamma(s+r)}, & s<-r\\ \psi(s), & -r\le s\le 0.\end{cases}$$

For $\psi^*C([-r, 0], \boldsymbol{R}^n)$, define ${}^r\psi^*\in C_\infty^\gamma$ as

$${}^r\psi^*(s)=\begin{cases}\psi^*(s), & -r\le s\le 0\\ \psi^*(-r)e^{-\gamma(s+r)}, & s<-r,\end{cases}$$

and define $L_2^*(t, \varphi, \psi^*)=L_2(t, \varphi, {}^r\psi^*)$. Since $L_2(t, \varphi, \psi)$ is a bounded linear functional with respect to ψ, there exists a constant $K(t, \varphi)$ such that

$$|L_2(t, \varphi, \psi)|\le K(t, \varphi)|\psi|_{C_\infty^\gamma} \qquad \text{for } \psi\in C_\infty^\gamma,$$

and hence

$$
\begin{aligned}
|L_2^*(t, \varphi, \psi^*)| &\leq K(t, \varphi) |{}^r\psi^*|_{C_\infty^r} \\
&\leq K(t, \varphi) \max \{|\psi^*(-r)| e^{-\gamma r}, \sup_{-r \leq s \leq 0} |\psi^*(s) e^{\gamma s}|\} \\
&\leq K(t, \varphi) \max \{e^{-\gamma r}, 1\} \sup_{-r \leq s \leq 0} |\psi^*(s)|.
\end{aligned}
$$

This means $L_2^*(t, \varphi, \psi^*)$ is a bounded linear functional with respect to $\psi^* \in C([-r, 0], \boldsymbol{R}^n)$. Therefore there exists a matrix $\eta(t, \varphi, s)$ with elements of bounded variation in $s \in [-r, 0]$ such that

$$L_2^*(t, \varphi, \psi^*) = \int_{-r}^{0} d_s \eta(t, \varphi, s) \psi^*(s),$$

and hence

$$D_\varphi(t, \varphi, \psi) = L_1(t, \varphi, {}_r\psi) + \int_{-r}^{0} d_s \eta(t, \varphi, s) \psi(s).$$

Define $A(t, \varphi) = \eta(t, \varphi, 0) - \eta(t, \varphi, 0^-)$, $r(t, \varphi, \alpha) = \mathrm{Var}_{s \in [-\alpha, 0]} \eta^*(t, \varphi, s)$, where $\eta^*(t, \varphi, s)$ is identical to $\eta(t, \varphi, s)$ on $[-r, 0]$ and $\eta^*(t, \varphi, 0) = \eta(t, \varphi, 0^-)$. If A, r satisfy the conditions in Definition 1, then $(d/dt) D(t, x_t) = f(t, x_t)$ is a NFDE.

If $D(t, \varphi) = \varphi(0)$ and $f\colon \Omega \to \boldsymbol{R}^n$, $\Omega \subseteq \boldsymbol{R} \times B$, then we get retarded functional differential equations with infinite delay discussed in [1], [6].

§ 3. The fundamental theory

Theorem 1. *Suppose Ω is an open set of $\boldsymbol{R} \times B$. Then for any $(\sigma, \varphi) \in \Omega$ there is a solution of NFDE (D, f, Ω) through (σ, φ).*

Proof. For $\alpha > 0$, $\beta > 0$, define $A(\alpha, \beta) = \{z \in C((-\infty, \alpha], \boldsymbol{R}^n);\ z(s) = 0,\ s \leq 0,\ |z(t)| \leq \beta,\ t \in [0, \alpha]\}$. Obviously, $A(\alpha, \beta)$ is a bounded, closed, convex subset of $BC((-\infty, \alpha], \boldsymbol{R}^n)$ (the space of bounded and continuous functions with the sup-norm $\|\cdot\|$).

We define two operators on $A(\alpha, \beta)$:

$$
(4) \qquad S\colon \begin{cases} Sz(t) = 0, & -\infty < t \leq 0 \\ A(\sigma + t, \tilde{\varphi}_t)(Sz)(t) = -L(\sigma + t, \tilde{\varphi}_t, z_t) - g(\sigma + t, \tilde{\varphi}_t, z_t) \\ \qquad + D(\sigma, \varphi) - D(\sigma + t, \tilde{\varphi}_t), & 0 \leq t \leq \alpha, \end{cases}
$$

and

$$
(5) \qquad U\colon \begin{cases} Uz(t) = 0, & -\infty < t \leq 0, \\ A(\sigma + t, \tilde{\varphi}_t)(Uz)(t) = \displaystyle\int_0^t f(\sigma + s, \tilde{\varphi}_s + z_s) ds, & 0 \leq t \leq \alpha \end{cases}
$$

under (1), where $\tilde{\varphi} \in F_\alpha(\varphi)$ with $\tilde{\varphi}(t) = \varphi(0)$ for $t > 0$, and

$$g(\sigma, \varphi, \psi)=D(\sigma, \varphi+\psi)-D(\sigma, \varphi)-D_\varphi(\sigma, \varphi)\psi. \tag{6}$$

1° By (2) and the continuity of f, D, D_φ, there are $\beta_0>0$ and a positive function $\alpha_1(\beta)$ defined for $0<\beta<\beta_0$ such that for $0<\beta<\beta_0$ and $0<\alpha<\alpha_1(\beta)$, $S+U$ maps $A(\alpha, \beta)$ into itself.

2° S is a contraction on $A(\alpha, \beta)$ for suitable α, β.

By the continuity of D_φ, for any $\varepsilon>0$, there are $\beta(\varepsilon)\in(0, \beta_0)$, $\alpha(\varepsilon)\in(0, \alpha_1(\beta(\varepsilon)))$ such that for $y, z\in A(\alpha(\varepsilon), \beta(\varepsilon))$, $t\in[0, \alpha(\varepsilon)]$,

$$|g(\sigma+t, \tilde{\varphi}_t, z_t)-g(\sigma+t, \tilde{\varphi}_t, y_t)|\leq\varepsilon|z_t-y_t|_B.$$

Therefore, for $0<\beta\leq\beta(\varepsilon)$, $0<\alpha\leq\alpha(\varepsilon)$, we have

$$\begin{aligned}
&\|Sz-Sy\| \\
&\quad\leq \sup_{0\leq t\leq\alpha}\{|A^{-1}(\sigma+t, \tilde{\varphi}_t)|[|L(\sigma+t, \tilde{\varphi}_t, y_t-z_t)|+|g(\sigma+t, \tilde{\varphi}_t, z_t)-g(\sigma+t, \tilde{\varphi}_t, y_t)|]\} \\
&\quad\leq \sup_{0\leq t\leq\alpha}|A^{-1}(\alpha+t, \tilde{\varphi}_t)|[r(\sigma+t, \tilde{\varphi}_t, t)+\varepsilon]|z_t-y_t|_B \\
&\quad\leq \sup_{0\leq t\leq\alpha}|A^{-1}(\sigma+t, \tilde{\varphi}_t)|K_1(t)[r(\sigma+t, \tilde{\varphi}_t, t)+\varepsilon]\|z-y\|.
\end{aligned}$$

Therefore, for a constant $k\in(0, 1)$, there are $0<\beta_2<\beta_0$ and a function $\alpha_2(\beta)$ defined on $[0, \beta_2]$, $\alpha_2(\beta)<\alpha_1(\beta)$ such that for $0<\beta<\beta_2$, $0<\alpha<\alpha_2(\beta)$, $z, y\in A(\alpha, \beta)$, we have

$$\|Sz-Sy\|\leq k\|z-y\|.$$

3° U is completely continuous on $A(\alpha, \beta)$ for $0<\beta<\beta_2$, $0<\alpha<\alpha_2(\beta)$.

For any $B\subseteq A(\alpha, \beta)$, $z\in B$, $0\leq t$, $\tau\leq\alpha$,

$$\begin{aligned}
&|Uz(t)-Uz(\tau)| \\
&\quad\leq\left|A^{-1}(\sigma+t, \tilde{\varphi}_t)\int_0^t f(\sigma+s, \tilde{\varphi}_s+z_s)ds-A^{-1}(\sigma+\tau, \tilde{\varphi}_\tau)\int_0^\tau f(\sigma+s, \tilde{\varphi}_s+z_s)ds\right| \\
&\quad\leq\left|A^{-1}(\sigma+t, \tilde{\varphi}_t)\int_\tau^t f(\sigma+s, \tilde{\varphi}_s+z_s)ds\right|+|A^{-1}(\sigma+t, \tilde{\varphi}_t)-A^{-1}(\sigma+\tau, \tilde{\varphi}_\tau)| \\
&\qquad\qquad\times\left|\int_0^\tau f(\sigma+s, \tilde{\varphi}_s+z_s)ds\right| \\
&\quad\leq N|t-\tau|+N|A^{-1}(\sigma+t, \tilde{\varphi}_t)-A^{-1}(\sigma+\tau, \tilde{\varphi}_\tau)|,
\end{aligned}$$

where N is a positive constant (by the continuity of A^{-1} and f, for sufficiently small α, β, we can find such an N). So, UB is uniformly bounded and equicontinuous, and hence UB is precompact by Ascoli's theorem. This implies U is completely continuous on $A(\alpha, \beta)$.

Obviously, by 2°, 3°, $S+U$ is an α-contraction on $A(\alpha, \beta)$. By Darbo's theorem (refer to Theorem 6.3 [2, p. 98]), $S+U$ has a fixed point.

Since

$$A(\sigma+t, \tilde{\varphi}_t)(Sz+Uz)(t)$$
$$=\int_0^t f(\sigma+s, \tilde{\varphi}_s+z_s)ds+D(\sigma, \varphi)-D(\sigma+t, \tilde{\varphi}_t+z_t)+A(\sigma+t, \tilde{\varphi}_t)z(t),$$

the integral equation

$$\begin{cases} D(\sigma+t, \tilde{\varphi}_t+z_t)-D(\sigma, \varphi)=\int_0^t f(\sigma+s, z_s+\tilde{\varphi}_s)ds \\ z_0=0 \end{cases}$$

has a continuous solution. This implies there is a solution of (3) through (σ, φ).

Theorem 2. *If there is a constant $L>0$ such that $|f(t, \varphi)-f(t, \psi)|\leq L|\varphi-\psi|_B$ for (t, φ), $(t, \psi) \in \Omega$, then for any $(\sigma, \varphi) \in \Omega$, there is a unique solution of* (3) *through* (σ, φ).

Proof. It is sufficient to prove $S+U$ has a unique fixed point in $A(\alpha, \beta)$. Suppose there are $z_1, z_2 \in A(\alpha, \beta)$ such that $z_i=(S+U)z_i$ $(i=1, 2)$. Then by 2°, we have

$$\sup_{0\leq s\leq t} |Sz_1(s)-Sz_2(s)|\leq k \sup_{0\leq s\leq t} |z_1(s)-z_2(s)|$$

and

$$|Uz_1(t)-Uz_2(t)|\leq|A^{-1}(\sigma+t, \tilde{\varphi}_t)|\left|\int_0^t [f(\sigma+s, \tilde{\varphi}_s+z_{1s})-f(\sigma+s, \tilde{\varphi}_s+z_{2s})]ds\right|$$
$$\leq|A^{-1}(\sigma+t, \tilde{\varphi}_t)|L\int_0^t |z_{1s}-z_{2s}|_B ds$$
$$\leq|A^{-1}(\sigma+t, \tilde{\varphi}_t)|L\int_0^t K_1(s) \sup_{0\leq\theta\leq s} |z_1(\theta)-z_2(\theta)|ds$$
$$\leq LL_1\int_0^t \sup_{0\leq\theta\leq s} |z_1(\theta)-z_2(\theta)|ds,$$

where $L_1=\sup_{0\leq t\leq\alpha}|A^{-1}(\sigma+t, \tilde{\varphi}_t)| \sup_{0\leq t\leq\alpha} K_1(t)$.

Putting $m(t)=\sup_{0\leq s\leq t}|z_1(s)-z_2(s)|$, we have

$$m(t)\leq km(t)+LL_1\int_0^t m(s)\,ds,$$

and

$$m(t)\leq\frac{LL_1}{1-k}\int_0^t m(s)\,ds.$$

By Gronwall's inequality, $m(t)=0$ for $0\leq t\leq\alpha$. Therefore, $z_1=z_2$.

Theorem 3. *Consider a NFDE(D, f, Ω), and suppose that for any closed and bounded set W with an $O(W, \delta)\subseteq\Omega$, f maps W into a bounded set in $\mathbf{R}^n$, $D(t, \varphi)$,*

$D_\varphi(t, \varphi)$ are uniformly continuous on W, and there are constants $s_0>0$, $N>0$ and continuous functions $r(s)$, $N_1(s)$ on $[0, s_0]$ with $r(0)=0$ such that

$$|A^{-1}(t, \varphi)|\leq N, \qquad |L(t, \varphi, \psi)|\leq r(s)|\psi|_{[-s,0]}+N_1(s)|\psi|_s$$

in (1) *if $(t, \varphi)\in W$, $s\in[0, s_0]$, where $O(W, \delta)$ denotes the δ-neighborhood of W,*

$$|\varphi|_{[-s,0]}=\inf_{\hat\varphi\in\hat B}\{\sup_{-s\leq\theta\leq 0}|\hat\psi(\theta)|;\ \psi=\varphi\} \qquad \textit{for } \varphi\in B.$$

If x is a noncontinuable solution of (3) *on $(-\infty, b)$, then for any closed and bounded set W_0 with an $O(W_0, \delta)\subseteq\Omega$ there is a sequence $t_n\to b^-$ as $n\to\infty$ such that $(t_n, x_{t_n})\notin W_0$.*

Proof. It is sufficient to prove that there is a $t'\in[\sigma, b)$ such that $(t', x_{t'})\notin W_0$. We may assume $b<+\infty$. Let $V=\{(t, x_t);\ t\in[\sigma, b)\}$. Obviously, it is sufficient to prove V can not be a bounded set or does not have a δ-neighborhood in Ω.

Suppose that V is bounded and has a δ-neighborhood in Ω. If $x(t)$ is uniformly continuous on $[\sigma, b)$, then $x(t)$ is continuous on $[\sigma, b]$ by setting $x(b)=\lim_{t\to b^-}x(t)$, and $(t, x_t)\to(b, x_b)$ as $t\to b^-$ by (α_5), and hence $x(t)$ is continuable beyond b (by Theorem 1), which is a contradiction.

So, $x(t)$ is not uniformly continuous on $[\sigma, b)$. Given $\beta\in(0, \delta)$, for $W=\overline{O(V, \beta)}$, choose N, $r(s)$, $N_1(s)$, $s_0>0$ as in the assumption, and choose $0<s<s_0$ so that

$$|r(s)|<1/7N. \tag{7}$$

There are $\varepsilon_0>0$, $t_k\in[b-s, b)$, $\Delta_k>0$, $\Delta_k\to 0$ as $k\to\infty$ such that

$$|x(t_k)-x(t_k-\Delta_k)|\geq\varepsilon_0 \qquad (k=1, 2, 3, \cdots). \tag{8}$$

Let $s_k=\inf\{t\in[b-s, b);\ |x(t)-x(t-\Delta_k)|\geq\beta^*\}$, where

$$K_1^*=\sup_{0\leq s\leq b-\sigma}K_1(s), \qquad M_1^*=\sup_{0\leq s\leq b-\sigma}M_1(s), \qquad \beta^*=\min\{\varepsilon_0, \beta/4K_1^*\}. \tag{9}$$

By the uniform continuity of $x(t)$ on any closed subset of (σ, b) and the uniform continuity of $D(t, \varphi)$ on $O(V, \beta)$, for any

$$\varepsilon_1<\min\{\beta^*/7N_1(s)NM_1^*K_1^*,\ \beta^*,\ \beta^*/7N\} \tag{10}$$

there exists $\Delta^*>0$ such that

$$|x(t')-x(t'')|<\varepsilon_1 \qquad \text{for } \sigma\leq t'<t''<t'+\Delta^*\leq b-s/2, \tag{11}$$

and

$$|D(t', \varphi')-D(t'', \varphi')|<\varepsilon_1 \qquad \text{for } (t', \varphi'), (t'', \varphi')\in O(V, \beta),\ \ |t'-t''|<\Delta^*. \tag{12}$$

Choose $n_0>0$ such that for $k\geq n_0$,

$$\Delta_k<\min(\Delta^*, s/2),$$

and

$$M_1^*|x_{\sigma+\Delta_k}-x_\sigma|_B<\beta/4. \tag{13}$$

Then we have

$$\begin{aligned}|x_{s_k}-x_{s_k-\Delta_k}|_B&\leq K_1(s_k-\Delta_k-\sigma)\cdot\sup_{-(s_k-\Delta_k-\sigma)\leq\theta\leq 0}|x(s_k+\theta)-x(s_k-\Delta_k+\theta)|\\&\qquad+M_1(s_k-\Delta_k-\sigma)|x_{\sigma+\Delta_k}-x_\sigma|_B\\&\leq K_1^*\beta^*+\beta/4\leq\beta/2\end{aligned} \tag{14}$$

and

$$\begin{aligned}&|L(s_k, x_{s_k}, x_{s_k}-x_{s_k-\Delta_k})|\\&\quad\leq r(s)|x_{s_k}-x_{s_k-\Delta_k}|_{[-s,0]}+N_1(s)|x_{s_k}-x_{s_k-\Delta_k}|_s\\&\quad\leq r(s)\sup_{-s\leq\theta\leq 0}|x(s_k+\theta)-x(s_k-\Delta_k+\theta)|+N_1(s)|\tau^s(x_{s_k-s}-x_{s_k-\Delta_k-s})|_s\\&\quad\leq r(s)\sup_{-s\leq\theta\leq 0}|x(s_k+\theta)-x(s_k-\Delta_k+\theta)|+N_1(s)M_1(s)|x_{s_k-s}-x_{s_k-\Delta_k-s}|_B\\&\quad\leq\beta^*/7N+N_1(s)M_1(s)K_1(s_k-\Delta_k-s-\sigma)\sup_{\sigma\leq\theta\leq s_k-\Delta_k-s}|x(\Delta_k+\theta)-x(\theta)|\\&\qquad+N_1(s)M_1(s)M_1(s_k-s-\Delta_k-\sigma)|x_{\Delta_k+\sigma}-x_\sigma|_B\\&\quad\leq\beta^*/7N+N_1(s)M_1^*K_1^*\varepsilon_1+N_1(s)M_1^{*2}|x_{\sigma+\Delta_k}-x_\sigma|_B.\end{aligned} \tag{15}$$

By the uniform continuity of D_φ on $\overline{O(V,\beta)}$ and (14), there is a nonnegative continuous function $\Sigma(\beta)$, $\Sigma(0)=0$, such that

$$\begin{aligned}|g(s_k, x_{s_k}, x_{s_k-\Delta_k}-x_{s_k})|&\leq\Sigma(\beta)|x_{s_k}-x_{s_k-\Delta_k}|_B\\&\leq\Sigma(\beta)K_1^*\beta^*+\Sigma(\beta)M_1^*|x_{\sigma+\Delta_k}-x_\sigma|_B.\end{aligned} \tag{16}$$

Since

$$\begin{aligned}&D(s_k, x_{s_k})-D(s_k-\Delta_k, x_{s_k-\Delta_k})\\&\quad=D(s_k, x_{s_k-\Delta_k})-D(s_k-\Delta_k, x_{s_k-\Delta_k})-A(s_k, x_{s_k})[x(s_k-\Delta_k)-x(s_k)]\\&\qquad-L(s_k, x_{s_k}, x_{s_k-\Delta_k}-x_{s_k})-g(s_k, x_{s_k}, x_{s_k-\Delta_k}-x_{s_k}),\end{aligned}$$

we have by (10), (12), (15), (16),

$$\begin{aligned}&|x(s_k)-x(s_k-\Delta_k)|\\&\quad\leq N[|L(s_k, x_{s_k}, x_{s_k-\Delta_k}-x_{s_k})|+|g(s_k, x_{s_k}, x_{s_k-\Delta_k}-x_{s_k})|\\&\qquad+|D(s_k, x_{s_k-\Delta_k})-D(s_k-\Delta_k, x_{s_k-\Delta_k})|+|D(s_k, x_{s_k})-D(s_k-\Delta_k, x_{s_k-\Delta_k})|]\\&\quad\leq N[\beta^*/7N+N_1(s)M_1^*K_1^*\varepsilon_1+N_1(s)M_1^{*2}|x_{\sigma+\Delta_k}-x_\sigma|_B+\Sigma(\beta)K_1^*\beta^*\\&\qquad+\Sigma(\beta)M_1^*|x_{\sigma+\Delta_k}-x_\sigma|_B+\varepsilon_1+M^0\Delta_k]\end{aligned}$$

for $k \geq n_0$, where $M^0 = \sup_{(t,\varphi)\in V} |f(t, \varphi)|$.

Choosing $n_1 > 0$, $\beta_0^* > 0$ so that for any $k \geq n_1$, $\beta \leq \beta_0^*$,

$$\Sigma(\beta)K_1^* < 1/7N,$$
$$\Sigma(\beta)M_1^* |x_{\sigma+\Delta_k} - x_\sigma|_B < \beta^*/7N,$$
$$M^0\Delta_k < \beta^*/7N, \quad N_1(s)M_1^{*2}|x_{\sigma+\Delta_k} - x_\sigma|_B < \beta^*/7N.$$

We have for $k \geq \max(n_0, n_1)$, $0 < \beta < \beta_0^*$,

$$|x(s_k) - x(s_k - \Delta_k)| < \beta^*$$

which is contrary to the definition of s_k. This implies V can not be a bounded set with a neighborhood in Ω. The theorem is completely proved.

Lemma 1. *Suppose that Γ is a closed, bounded, convex set of a Banach space, that Λ^* is a subset of another Banach space and that $T = S + U$: $\Gamma \times \Lambda^* \to \Gamma$ satisfies the following hypotheses*:

(i) *for some $\lambda_0 \in \Lambda^*$ the equation $x = T(x, \lambda_0)$ has a unique solution $x(\lambda_0)$ in Γ,*

(ii) *$U(x, \lambda)$ is continuous in $(x, \lambda) \in \Gamma \times \Lambda^*$, and for each compact set $\Lambda' \subseteq \Lambda^*$, $U(\Gamma, \Lambda')$ is precompact,*

(iii) *$S(\cdot, \lambda)$ is a contraction for each $\lambda \in \Lambda^*$ and $S(x, \lambda)$ is continuous at λ_0 uniformly for $x \in \Gamma$.*

Then the solution $x(\lambda)$ of equation $x = T(x, \lambda)$ is continuous at λ_0.

For the proof, we refer to [2, Chapter 12, Corollary 2.2].

Theorem 4. *Suppose Ω is an open set of $\boldsymbol{R} \times B$, Λ is a subset of another Banach space X and D, f: $\Omega \times \Lambda \to \boldsymbol{R}^n$ satisfies*

(i) *$f(t, \varphi, \lambda)$, $D(t, \varphi, \lambda)$, $D_\varphi(t, \varphi, \lambda)$ are continuous in $(t, \varphi, \lambda) \in \Omega \times \Lambda$,*

(ii) *there are a $\beta_0 > 0$ and a nonnegative continuous function $r(t, \varphi, \beta, \lambda)$ on $\Omega \times [0, \beta_0] \times \Lambda$ such that $r(t, \varphi, 0, \lambda) = 0$ for $(t, \varphi, \lambda) \in \Omega \times \Lambda$, and $D_\varphi(t, \varphi, \lambda)\psi = A(t, \varphi, \lambda)\psi(0) + L(t, \varphi, \psi, \lambda)$, where $A(t, \varphi, \lambda)$, $A^{-1}(t, \varphi, \lambda)$ are continuous, and*

$$|L(t, \varphi, \psi, \lambda)| \leq r(t, \varphi, \beta, \lambda)|\psi|_B \quad \textit{if } \tilde{\psi}(\theta) = 0 \quad \textit{for } \theta \leq -\beta_0 < 0,$$

(iii) *for some $\lambda_0 \in \Lambda$, there is a unique solution of NFDE $(D(\cdot, \lambda_0), f(\cdot, \lambda_0), \Omega)$ through $(\sigma_0, \psi_0) \in \Omega$ which exists on $(-\infty, b]$, $b > \sigma_0$.*

Then, there exists a neighborhood $N(\sigma_0, \varphi_0, \lambda_0)$ of $(\sigma_0, \varphi_0, \lambda_0)$ such that for any $(\sigma', \varphi', \lambda') \in N(\sigma_0, \varphi_0, \lambda_0)$, the solution $x(t; \sigma', \varphi', \lambda')$ of NFDE$(D(\cdot, \lambda'), f(\cdot, \lambda'), \Omega)$ through (σ', φ') exists on $(-\infty, b]$ and is continuous in $(t, \sigma', \varphi', \lambda')$ at $(t, \sigma_0, \varphi_0, \lambda_0)$ for $t \in [\sigma', b]$.

Proof. Using the similar method used in the proof of Theorem 1, we can prove that there are $\alpha > 0$, $\beta > 0$, a neighborhood $N(\sigma_0, \varphi_0)$ of (σ_0, φ_0) and a neighborhood $O(\lambda_0)$ of λ_0 in X such that for $(\sigma', \varphi', \lambda') \in N(\sigma_0, \varphi_0)\times O(\lambda_0)$, $x(t, \sigma', \varphi', \lambda')$ exists on $(-\infty, \sigma+\alpha)$ and $S(\sigma', \varphi', \lambda')+U(\sigma', \varphi', \lambda')$: $A(\alpha, \beta)\to A(\alpha, \beta)$.

It is easy to prove that for $\Lambda^*=N(\sigma_0, \varphi_0)\times(\Lambda\cap O(\lambda_0))$ and $\Gamma=A(\alpha, \beta)$, all the conditions of Lemma 1 are satisfied. So the solution of NFDE $(D(\cdot, \lambda'), f(\cdot, \lambda'), \Omega)$ through (σ', φ') is continuous at $(\lambda_0, \sigma_0, \varphi_0, t)$ on $[\sigma', \sigma'+\alpha]$.

By the compactness of $\{(t, x_t(\sigma_0, \varphi_0, \lambda_0)); t \in [\sigma_0, b]\}$ we can completely prove the theorem using the finite covering theorem.

§ 4. The continuation of quasi-linear system

Let Ω be an open set in $\boldsymbol{R}\times B$, $D, f \in C(\Omega, \boldsymbol{R}^n)$, $D(t, \varphi)=\varphi(0)-T(t)\varphi$, and let $T(t)$ be a bounded linear operator. If there exist continuous maps r, N: $\boldsymbol{R}\times\boldsymbol{R}^+\to\boldsymbol{R}^+$ with $r(t, 0)=0$ such that for any $(t, \varphi, s) \in \Omega\times\boldsymbol{R}^+$, $t_0 \in \boldsymbol{R}$

$$|T(t)\varphi|\le r(t, s)|\varphi|_{[-s,0]}+N(t, s)|\varphi|_s, \tag{17}$$

$$|||T(t)-T(t_0)|||\longrightarrow 0 \quad \text{as} \quad t\longrightarrow t_0, \tag{18}$$

where $|||\cdot|||$ denotes the operator norm, then we say

$$\frac{d}{dt}D(t, x_t)=f(t, x_t) \tag{19}$$

is a quasi-linear neutral functional differential equation with infinite delay. We denote (19) by QLNFDE (D, f, Ω).

The following lemma can be proved by the same way as in [1, Lemma 2.1].

Lemma 2. *Let* $F^C_{A,0}=\{x \in F_{A,0}: |x(t)|\le C,\ t \in [0, A]\}$, *and let* $G\subseteq F^C_{A,0}$. *If* $A<+\infty$, $\{x_0: x \in G\}$ *is a compact set in* B *and* $x(t)$, $x \in G$, is *isequicontinuous in* $t \in [0, A]$, *then* $\Gamma_0=\{x_t;\ x \in G,\ t \in [0, A]\}$ *is a compact set in* B *and* x_t, $x \in G$, *is equicontinuous in* $t \in [0, A]$.

Theorem 5. *Suppose that for any* $t_0 \in \boldsymbol{R}$, $\alpha\ge 0$, *if a function* x *defined on* $(-\infty, \alpha+t_0)$ *with* $x_{t_0} \in B$ *is continuous on* $[t_0, \alpha+t_0)$, *and if there exists a sequence* $t_k\to(\alpha+t_0)^-$ *such that* $\sup_k|x_{t_k}|_B<+\infty$, *then there is an* $L(x)>0$ *such that* $|x(t)|\le L(x)$ *for* $t \in [t_0, \alpha+t_0)$.

Let $x(t)$ *be a noncontinuable solution on* $(-\infty, \delta)$ $(\delta>\sigma)$ *of QLNFDE* (D, f, Ω) *through* (σ, φ). *Then for any compact set* $W\subseteq\Omega$, *there is a* t_w *such that* $(t, x_t) \notin W$ *for* $t_w\le t<\delta$.

Proof. If the conclusion is false, then for some compact set W in Ω, there is a sequence $t_k\to\delta^-$ such that $(t_k, x_{t_k}) \in W$. Then since W is compact, $\delta<+\infty$ and $|x(t)|\le L(x)=C$ for $t \in [\sigma, \delta)$, where we note that

$$\sup_k |x_{t_k}|_B \leq M^* \qquad \text{for } M^* = \sup\{|\varphi|_B; (t, \varphi) \in W \text{ for a } t \in R\},$$

we may assume $(t_k, x_{t_k}) \to (\delta^-, \varphi) \in W \subseteq \Omega$ and there are $\varepsilon_0 > 0$, $M > 0$ such that $|f(s, \psi)| \leq M$ for $(s, \psi) \in O(W, \varepsilon_0)$.

If we can prove that x_t converges to the φ as $t \to \delta^-$, then clearly the solution $x(t)$ must be continuable beyond δ, which is contrary to the assumption. So we may assume there is a sequence $t'_k \to \delta^-$ and an $\varepsilon \in (0, \varepsilon_0)$ such that for $k = 1, 2, \cdots$, $t_k < t'_k$, $|x_{t_k} - x_{t'_k}|_B = \varepsilon$ and $|x_t - x_{t_k}|_B < \varepsilon$ for $t_k \leq t \leq t'_k$.

Define x^k: $x^k(t) = x(t + t_k)$ if $t \leq t'_k - t_k$, $= x(t'_k)$ if $t > t'_k - t_k$.

Obviously, $x^k \in F^C_{1,0}$. Let $G = \{\bar{\varphi}\} \cup (\bigcup_{k=1}^\infty \{x^k\})$, where $\bar{\varphi} \in F_{\infty,0}$ with $\bar{\varphi}(t) = \varphi(0)$ for $t > 0$. Then $\{x_0; x \in G\} = \{\varphi\} \cup (\bigcup_{k=1}^\infty \{x_{t_k}\})$ is compact. We want to prove $x^k(t)$ is equi-continuous in $t \in [0, 1]$. If this is not true, then there exist an $\varepsilon_1 > 0$ and sequences $\{t_{1k}\}$, $\{t_{2k}\}$ such that $t_{n_k} \leq t_{1k} < t_{2k} < t'_{n_k}$, $|x(t_{1k}) - x(t_{2k})| \geq \varepsilon_1$, where $\{t_{n_k}\}$, $\{t'_{n_k}\}$ are subsequences of $\{t_k\}$, $\{t'_k\}$.

On the other hand

$$\begin{aligned}
&|x(t_{1k}) - x(t_{2k})| \\
&\quad \leq \left| \int_{t_{1k}}^{t_{2k}} f(s, x_s) ds \right| + |T(t_{1k}) x_{t_{1k}} - T(t_{2k}) x_{t_{2k}}| \\
&\quad \leq M|t_{1k} - t_{2k}| + |||T(t_{1k}) - T(t_{2k})|||[|x_{t_k}|_B + \varepsilon] \\
&\qquad + r(t_{1k}, s)|x_{t_{1k}} - x_{t_{2k}}|_{[-s,0]} + N(t_{1k}, s)|x_{t_{1k}} - x_{t_{2k}}|_s \\
&\quad \leq M|t_{n_k} - t'_{n_k}| + |||T(t_{1k}) - T(t_{2k})|||(M^* + \varepsilon) \\
&\qquad + r(t_{1k}, s) \sup_{-s \leq \theta \leq 0} |x(t_{1k} + \theta) - x(t_{2k} + \theta)| + N(t_{1k}, s) M_1(s) |x_{t_{1k} - s} - x_{t_{2k} - s}|_B.
\end{aligned}$$

Choose $s > 0$ so that $|r(t, s)| < \varepsilon_1 / 8C$ for $t \in [\sigma, \delta]$. Since x_t is continuous at $\delta - s$, there exists an $n_0 > 0$ such that for $k \geq n_0$,

$$\sup_{0 \leq t \leq \delta} N(t, s) M_1(s) |x_{t_{1k} - s} - x_{t_{2k} - s}|_B < \varepsilon_1 / 4,$$

$$M|t'_{n_k} - t_{n_k}| < \varepsilon_1 / 4, \qquad |||T(t_{1k}) - T(t_{2k})||| < \varepsilon_1 / 4(M^* + \varepsilon).$$

So, for $k \geq n_0$, $|x(t_{1k}) - x(t_{2k})| < \varepsilon_1$. This is contrary to the definition of t_{1k}, t_{2k}. Then x^k is equi-continuous in $t \in [0, 1]$ and so by Lemma 2, $|x^k_0 - x^k_{t'_k - t_k}|_B = |x_{t_k} - x_{t'_k}|_B \to 0$ as $k \to \infty$. This is also a contradiction.

Theorem 6. *Suppose that the phase space satisfies*

(α_6) *for any $\alpha > 0$, $t_0 \in \mathbf{R}$ and a function x defined on $(-\infty, \alpha + t_0)$ with $x_{t_0} \in B$ and x being continuous on $[t_0, \alpha + t_0)$, if there exists a sequence $t_k \to (\alpha + t_0)^-$ such that x_{t_k} converges as $k \to \infty$, then $x(t)$ converges as $t \to (\alpha + t_0)^-$.*

Under the same conditions as in Theorem 5, if f maps any bounded closed set of Ω into a bounded set in $\mathbf{R}^n$, then for any bounded closed set W in Ω, there is a t_w such that $(t, x_t) \notin W$ for $t_w \leq t < \delta$.

Proof. We use the similar method as in ([1], Theorem 2.4) to prove the theorem.

If this is not true, that is, there exists $t_k \to \delta^-$ such that $(t_k, x_{t_k}) \in W$, then $x(t)$ is bounded by a constant C_0 on $[\sigma, \delta)$ and $\delta < +\infty$. We prove $\Gamma = \mathrm{Cl}\{(t, x_t);\ t \in [\sigma, \delta)\}$ is a bounded set in Ω.

If $\Gamma \subseteq \Omega$ but Γ is not bounded, then there exists a sequence $s_k \to \delta^-$ such that

$$s_{k-1} < t_k < s_k, \quad |x_{s_k}|_B = C \quad \text{and} \quad |x_t|_B < C \quad \text{for } t \in [t_k, s_k)$$

where

$$C = K_1^*(KC_1 + N_1^* M_1^* + 2C_0) + M_1^* C_1 + 1 + C_1 + K_1^* C_1,$$
$$K_1^* = \sup_{0 \le s \le 1} K_1(s), \qquad M_1^* = \sup_{0 \le s \le 1} M_1(s),$$
$$C_1 = \sup\{|\varphi|_B;\ (t, \varphi) \in W \text{ for some } t \in \boldsymbol{R}\},$$

and

$$N_1^* = \sup\{N(t, s);\ 0 \le t \le \delta,\ 0 \le s \le 1\}.$$

Define $\Gamma_0 = \mathrm{Cl}\{(t, x_t);\ t \in \bigcup_k [t_k, s_k]\} \subseteq \Gamma \subseteq \Omega$, then $|f(t, x_t)|$ is bounded by an M on Γ_0. Let $\beta_k = s_k - t_k$. From (19) we get

$$\begin{aligned}
|x_{s_k}|_B &\le K_1(s_k - t_k) \sup_{t_k \le \theta \le s_k} |x(\theta)| + M_1(s_k - t_k)|x_{t_k}|_B \\
&\le K_1(\beta_k)\{|x(t_k)| + M|s_k - t_k| + \sup_{t_k \le \theta \le s_k} |T(\theta)x_\theta - T(t_k)x_{t_k}|\} + M_1(\beta_k)|x_{t_k}|_B \\
&\le K_1(\beta_k)\{KC_1 + M|s_k - t_k|\} + K_1(\beta_k)\{\sup_{t_k \le \theta \le s_k} |[T(\theta) - T(t_k)]x_{t_k}| \\
&\qquad\qquad + \sup_{t_k \le \theta \le s_k} |T(\theta)(x_\theta - x_{t_k})|\} + M_1(\beta_k)C_1 \\
&\le K_1(\beta_k)[KC_1 + M(s_k - t_k)] + K_1(\beta_k) \sup_{t_k \le \theta \le s_k} |||T(\theta) - T(t_k)|||\, C_1 \\
&\quad + K_1(\beta_k) \sup_{t_k \le \theta \le s_k} [r(\theta, s)|x_\theta - x_{t_k}|_{[-s,0]} + N(\theta, s)|x_\theta - x_{t_k}|_s] + M_1(\beta_k)C_1 \\
&\le K_1(\beta_k)[KC_1 + M(s_k - t_k)] + K_1(\beta_k)C_1 \sup_{t_k \le \theta \le s_k} |||T(\theta) - T(t_k)||| \\
&\quad + K_1(\beta_k)[2C_0 + \sup_{t_k \le \theta \le s_k} N_1^* M_1(s)|x_{\theta-s} - x_{t_k-s}|_B] + M_1(\beta_k)C_1
\end{aligned}$$

where s is given so that $\sup_{0 \le t \le s} r(t, s) < 1$.

Choose $n_1 > 0$ so that $|x_{t_k-s} - x_{\theta-s}|_B \le 1$, $\beta_k \le 1$, $t_k - s \ge \sigma$, $|||T(\theta) - T(t_k)||| < 1$, $K_1^* M_1(s_k - t_k) < C_1 + 1$ for $k \ge n_1$, $\theta \in [t_k, s_k]$. Then for $k \ge n_1$, $|x_{s_k}|_B < C$ which yields a contradiction.

Let $\Gamma \not\subseteq \Omega$. Since $\{(t, x_t);\ t \in [\sigma, \delta)\} \subseteq \Omega$, there exists a sequence $(\sigma_k, x_{\sigma_k}) \to (\delta, \varphi)$ as $k \to \infty$, $(\delta, \varphi) \in \Gamma \cap \partial\Omega$. By (α_6), x has a continuous extension beyond δ. So, Γ is compact and therefore for the compact set $W_0 = W \cap \Gamma$ in Ω, there is a t_{w_0} such that $(t, x_t) \notin W_0$ for $t \in [t_{w_0}, \delta)$ by Theorem 5. This implies $(t, x_t) \notin W$ for $t \in [t_{w_0}, \delta)$. This contradicts the existence of the sequence $\{t_k\}$.

So, if the conclusion of the theorem is not true, then Γ is a bounded closed set in Ω, we can prove $x(t)$ is uniformly continuous on $[\sigma, \delta)$ by the method used in the proof of Theorem 3. Therefore Γ is a compact set in Ω. Repeating the same argument as above yields a contradiction. This proves completely the theorem.

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